Assignment 4

Hand in no. 1, 5, 6 and 8 by October 3, 2019.

1. Prove Hólder's Inequality in vector form: For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,
$$|\mathbf{a} \cdot \mathbf{b}| \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q} .$$

2. A quick proof of Hölder's Inequality consists of two steps: First, assuming $\|f\|_p = \|g\|_p = 1$ and integrate Young's Inequality. Next, observe that $f/\|f\|_p$ satisfies the first step. Can you find any disadvantage of this approach?

3. Prove the generalized Hölder's Inequality: For $f_1, f_2, \cdots, f_n \in R[a,b]$,
$$\int_a^b |f_1 f_2 \cdots f_n| dx \leq \left(\int_a^b |f_1|^{p_1} \right)^{1/p_1} \left(\int_a^b |f_2|^{p_2} \right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n} \right)^{1/p_n} ,$$
where
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 .$$

4. Establish the inequality, for $f \in R[a,b]$,
$$\int_a^b |f| \, dx \leq (b-a)^{1/q} \left(\int_a^b |f|^p \, dx \right)^{1/p} , \quad 1/p + 1/q = 1, \; p > 1 .$$

5. Establish the inequality, for $f \in R[a,b]$, $\|f\|_{p_1} \leq C \|f\|_{p_2}$ when $1 \leq p_1 < p_2$.

6. Show that there is no constant C such that $\|f\|_2 \leq C\|f\|_1$ for all $f \in C[0,1]$.

7. Show that $\|a\|_p$ is no longer a norm on $\mathbb{R}^n$ for $p \in (0,1)$.

8. Show that $\|f\|_p$ is no longer a norm on $C[0,1]$ for $p \in (0,1)$.

9. Optional. Show that any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are equivalent, that is, there exists two constants C_1, C_2 such that $\|x\|_1 \leq C_1 \|x\|_1$ and $\|x\|_2 \leq C_2 \|x\|_1$ for all $x \in \mathbb{R}^n$. Hint: It suffices to show every norm is equivalent to the Euclidean norm.

10. Let l^p consist of all sequences $\{a_n\}$ satisfying $\sum_n |a_n|^p < \infty$. Show that
$$\|a\|_p = \left(\sum_n |a_n|^p \right)^{1/p} ,$$
defines a norm on $l^p, 1 \leq p < \infty$. Propose a definition for the normed space l^∞.

11. Define d on $\mathbb{Z} \times \mathbb{Z}$ by $d(n,m) = 2^{-d}$, where d is the largest power of 2 dividing $n - m \neq 0$ and set $d(n,n) = 0$. Verify that d defines a metric on $\mathbb{Z}$.